Homotopy type theory and the vertical unity of concepts in mathematics

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February 2016

Abstract

The mathematician Alexander Borovik speaks of the importance of the 'vertical unity' of mathematics. By this he means to draw our attention to the fact that many sophisticated mathematical concepts, even those introduced at the cutting-edge of research, have their roots in our most basic conceptualisations of the world. If this is so, we might expect any truly fundamental mathematical language to detect such structural commonalities.

It is reasonable to suppose then that the lack of philosophical interest in such vertical unity is related to the prominence given by philosophers to languages which do not express well such relations. In this chapter, I suggest that we look beyond set theory to the newly emerging homotopy type theory, which makes plain what there is in common between very simple aspects of logic, arithmetic and geometry and much more sophisticated concepts.

Key words: concept; homotopy type theory; vertical unity; arithmetic; space; duality

1 Introduction

Studying the articulation of mathematical concepts is an important task for philosophy, not least for making us lift our eyes above the level of proof to consider the larger enterprise of doing mathematics. Where so much of the Anglophone philosophical literature has dwelt on mathematical activity as essentially a matter of proof construction, the larger top-down view of mathematics as a tradition of intellectual enquiry, rooted many millennia in the past, is typically neglected. Even Lakatos, that pioneer of the study of conceptual change, took proof construction and criticism to be the central motors of conceptual development, rather than tackle larger strategic units of research. My earliest published works in philosophy (Corfield 1997, 1998) aimed at showing the limitations of this outlook by posing and attempting to answer the question of how we might understand larger-scale activity, such as the mathematical equivalents of the 'research programmes' Lakatos himself had studied in the field of natural science.

By the time I came to write *The Importance of Mathematical Conceptualisation* (Corfield 2001) it was with a view to direct attention towards yet higher levels of research activity. My examination of 'mathematical research programmes' had led me to see the enormously intricate patterns of lines of influence between mathematics generated in different fields. I chose then to take up as a case study forms of persuasive argument for the value of the concept of 'groupoid', since there were multiple lines of support for this concept from clusters of researchers working in different fields. There was also a countervailing current aiming to deny this value, which maintained that groupoids added little or nothing not already to be found in the concept of 'group'. Here then was to be found a debate not as to the correctness of pieces of mathematical reasoning, but one which concerned the worth of a certain extension of an existing concept. Groupoids, it was claimed by their advocates, had certain key advantages over groups when it came to capturing aspects of symmetry.

In the years since I wrote that piece, groupoids and their higher cousins have been proposed as the building blocks of a new foundational programme for mathematics, known as *homotopy type theory*, one which sees sets demoted from their dominant position to become a lowly component of a larger picture. If things work out well for this programme, it will be a great vindication of the claims made by those who glimpsed the potency of the groupoid concept, even if they never quite envisaged this lofty new role.

In this chapter I want to develop two theses:

1. This new foundational language, homotopy type theory, provides an important perspective from which to understand varieties of mathematical concept.

2. Mathematics displays *vertical unity*, so that concepts met in elementary and in research level mathematics are related.

Thesis 1 requires enough prefatory remarks on what is intended here by 'varieties of mathematical concept' to warrant its own section, but I can quickly sketch now what is meant by Thesis 2. *Vertical unity* is a term introduced by the mathematician Alexander Borovik to indicate that while the continued unfolding of mathematics at the research front appears to give rise to fiendishly complicated ideas, the products of this research in fact never completely depart from the kinds of concepts accessible to those with little mathematical training. This unity, Borovik claims, means that " 'recreational', 'elementary', 'undergraduate' and 'research' mathematics are no more than artificial subdivisions of a single continuous spectrum" (Borovik 2005, p. 1). Where we may take justifiable delight in the 'horizontal' unity provided by far-flung analogies between apparently different fields,

> [t]he vertical unity of mathematics, with many simple ideas and tricks working both at the most elementary and at rather sophisticated levels, is not so frequently discussed – although it appears to be highly relevant to the very essence of mathematics education. (Borovik 2005, p. 10)

I cannot speak for any other than the philosophical literature in this regard, but already this lack of attention there is interesting. I suspect that the standard representation of mathematics through the filter of first-order logic and set theory is to blame. By the time one has rendered conceptually similar pieces of mathematics into set theory, little or no sign remains of their resemblance. We will see then that Thesis 1 is highly relevant to Thesis 2, since homotopy type theory offers a more faithful representation of mathematics, enough to allow commonality of structure to show through.

After the next section on varieties of concept, I first illustrate this vertical unity with a case which takes its starting point from a commonality found between elementary logic and elementary arithmetic. It transpires that the common structure found between these domains can be seen as arising from the first two steps in a hierarchy proposed by the homotopy type theory program. In section 3, I return to my groupoid case study to see what further elaborations of the concept have been made. From groupoids I then turn to the mathematical concept of space, and show again how it has stayed true to a basic idea of everyday map-making, representing a space through overlapping charts. I briefly also consider the concept of duality.

2 Varieties of Concept

The term 'concept' itself designates a concept of course, one for which we should not expect a neat, formal definition. It belongs in English to a cluster of terms, including 'idea' and 'notion'. Whether these are used as synonyms or as slight variants is not easy to settle. In my book (Corfield 2003) I tended to rely overly on the term 'idea', as acutely observed by the mathematician Michael Harris.

> Corfield uses the same word to designate what I am calling "ideas" ("the ideas in Hopf's 1942 paper", p. 200) as well as "Ideas" ("the idea of groups", p. 212) and something halfway between the two (the "idea" of decomposing representations into their irreducible components for a variety of purposes, p. 206). Elsewhere the word crops up in connection with what mathematicians often call "philosophy," as in the "Langlands philosophy" ("Kronecker's ideas" about divisibility, p. 202), and in many completely unrelated conections as well. Corfield proposes to resolve what he sees as an anomaly in Lakatos' methodology of scientific research programmes as applied to mathematics by a shift of perspective from seeing a mathematical theory as a collection of statements making truth claims, to seeing it as the clarification and elaboration of certain central ideas (p. 181)
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> He sees "a kind of creative vagueness to the central idea" in each of the four examples he offers to represent this shift of perspective; but on my count the ideas he chooses include two 'philosophies," one "Idea," and one which is neither of these. (Harris 2008, p.16)

It is not that I believe I would have helped my case greatly by mixing things up with the use of, say, 'concept'. It seems to me that little is gained by carving up this semantic space with different terms, for example, by providing distinct, precise definitions for 'concept', 'idea' and 'notion'. But, as Harris suggests, there does seem to be a difference between, on the one hand, a core constituent of the visionary outlook of a large scale research programme, like that of Langlands, and, on the other, some stably defined kind of entity, such as a group. There is also a difference between the 'concept of (a) group' and the 'concept of symmetry', even if groups are the tool of choice to understand symmetry.

What we most need is as accurate as possible a description of mathematics as a tradition of intellectual enquiry. Any distinctions we draw should follow those we find in such a description. As Dudley Shapere (1987) argued, philosophers should not look to produce descriptions of science by refining 'metascientific' terms, such as observation, confirmation and explanation, independently from science itself, while the scientists go about their business, finding out about electrons, genes and tectonic plates. What it is to be an observation, a confirmation or an explanation changes as science learns more about the world and learns how to learn more effectively about the world. Similarly, it is not for philosophers of mathematics to impose some preordained notion of mathematical concept onto an account of mathematics. Indeed, we should expect such a 'meta-mathematical' term to require modification as mathematics proceeds.

One way we might expect to see these changes is through our foundational languages, especially when such a change occurs because of the demands of the practice. Certainly one of the most important motivations for Frege's work on his *Begriffsschrift* was to allow for the isolation and formation of fruitful concepts (Tappenden 1995). The components of first-order logic – variables, function symbols, relation symbols, connectives and quantifiers – thus provide a range of tools. With the rise of set theory, to these were added the resources of the axioms of set formation. However, in recent decades doubts have grown as to how helpful these tools have proved to be in making sense of the details of mainstream mathematics. It has been a common observation that while everything is in principle expressible in set theory, to carry out such expression would provide no illumination.

With the emergence of a new foundation, homotopy type theory, there should be many opportunities to explore its potential for concept formation. Type theories have a large array of ingredients, from type formation rules, such as those for *identity* types and *inductive* types, to introduction and elimination rules for terms. We should be able to see then that some concepts emerge from the intrinsic components of the system, while others are merely expressible within it. Type theories have very close ties with category theory (Harper 2013), which has a greater claim to represent faithfully, if not all of mainstream mathematics, then certainly large parts, such as algebraic geometry and algebraic topology. Homotopy type theory takes this association further to higher category theory (Lurie 2009a). In addition, some type theories allow for the direct expression of 'modalities,' to be mentioned in section 5, which allow forms of variation, cohesion and smoothness to be expressed. There are also linear flavours of type theory, associated with linear algebra and quantum mechanics (Schreiber 2014). There will be intersting opportunities to find out which concepts are deeply ingrained in such type theories.

Since it is all too easy for philosophers to be seduced into believing that their preferred formalism is the source of all important answers, it will help to keep us honest to pay close attention to the best thinking about concepts in the philosophical literature, especially that tuned to scientific concepts. Mark Wilson in *Wandering Significance* (2006) presents a strong case that concepts are rarely given by neat necessary and sufficient conditions, and that they rarely map straightforwardly onto the world. Instead, concepts are patched together from varied parts. Different patches provide different 'directivities' as to application. We tend to cover up these differences, he claims, behind 'classical facades'. When we come to see through such facades, Wilson argues, we ought not retreat to what he calls a 'hazy holism'.

Many of Wilson's examples come from applied mathematics and engineering, where we hear of the wandering significance of hardness, force, weight and gearwheel. The imagery used for concept extension is mathematical: prolongation, lifting, pulling back. Mathematical concepts are included too, as we hear in his description of Dedekind's number theoretic analogy, the largest case study of chap. 4 of Corfield 2003. In future work I will explore whether we might take Wilson's mathematical imagery of concept extension more literally when applied in mathematics itself. After all, the terms he deploys to convey this imagery are generally spatial and may perhaps be better addressed by a foundational system tuned to spatial notions, such as homotopy type theory. But now we need to delve into a case study to start us off with examples of the kinds of material we need for the theses of this chapter. Let's then consider a simple analogy between logic and arithmetic.

3 Addition, Multiplication and Exponentiation

When I teach an introductory course on logic to philosophy students, I like to point out to them certain similarities between the truth tables for the logical connectives and some simple arithmetic operations, similarities which have been known since the nineteenth century. If we assign the values 1 to True and 0 to False, then forming the conjunction ("and") of two propositions, the resulting truth value is formed very much as a product of numbers chosen from $\{0, 1\}$ is formed:

- Unless both values are 1, the product will be 0.
- Unless both truth values are True, the truth value of the conjunction will be False.

It is natural for students then to wonder if the disjunction ("or") of two propositions corresponds to addition. Here things don't appear to work out precisely. It's fine in the three cases where at least one of the propositions is False, for example, 'False or True' is True, just as $0 + 1 = 1$. But in the case of 'True or True', we seem to be dealing with an addition capped at 1, something such as would result in a third glass when the contents of two other full or empty glasses are emptied into it.

Setting to one side for the moment this deviation, the next connective usually encountered is "implies". A very good way of conveying to students the meaning of an implication between two propositions, such as $A \to B$, is as an 'inference ticket', a process taking a guarantee for the truth of A into such a guarantee for B. Implications seem to act very much like functions. Denoting the conjunction of A and B by $A \wedge B$, we may further observe the analogy:

- $(A \wedge B) \to C$ is True if and only if $A \to (B \to C)$ is True.
- $c^{(a \times b)} = (c^b)^a$

We may reformulate the former as expressing the equivalence between deducing C from the premises A and B, and deducing B implies C from the premise A.

Now what accounts for this pleasant analogy? The idea of a proof of $A \to B$ as a function and its analogy to the arithmetic construction b^a should bring to mind the fact that the latter may also be interpreted as related to functions. Indeed, b^a counts the number of distinct functions from a set of size a to a set of size b, so that, for example, there are 9 distinct functions from a set of size 2 to one of size 3.

Now we can see a closer analogy between proposition and sets, first through a parallel between conjunction and product:

- Guarantees of the proposition $A \wedge B$ can be formed from pairing together a guarantee for A and one for B.
- Elements of the product of two sets $A \times B$ can be formed from pairing together an element of A and one of B.

And second, there's a parallel between implication and function space, or exponential object:

- Guarantees of the proposition $A \to B$ are ways of sending guarantees for A to guarantees for B.
- Elements of the function space of two sets B^A (or $Fun(A, B)$) are functions mapping elements of A to elements of B.

This approach to logic is very much in line with the tradition of Per Martin-Löf (1984), and in philosophy with the ideas of Michael Dummett (1991). What homotopy type theory[1] does is take observations such as these not merely as an analogy, but as the first steps of a series of levels. Propositions and sets are both considered to be simple kinds of *types*. In Martin-Löf style type theories, given a type and two elements of that type, $a, b : A$, we can form the type of identities between a and b, often denoted $Id_A(a, b)$. The innovation of homotopy type theory is that this identity type is not required to be a proposition – there may be a richer type of such identities. But we can see now how the levels bottom out. If you give me two expressions for elements of a given *set*, the answer to the question of whether or not they are the same is 'yes' or 'no'. We say that the identity type is a proposition. On the other hand, in the case of a type which is a proposition, were we to have constructed two elements, in this case proofs, and so know the proposition to be true, these elements could only be equal. Here we are not distinguishing between different proofs of a proposition, which has led some to call them 'mere propositions.'

So taking identity types of the terms of a type decreases the type's level, and this may be iterated until we reach a point where the type resembles a singleton point. For historical reasons connected to the branch of mathematics known as homotopy theory this level is designated -2. Mere propositions are thus of level -1, and sets of level 0. But we can find higher level types, ones for which there may be a *set* of identities between two terms, or a type of identities of yet higher level. Take the type of finite sets. For any two elements, that is, finite sets, an element of their identity type is any map which forms a one-to-one correspondence between the sets. In the case of large equally sized sets, there

[1] The term 'Univalent Foundations' is sometimes used synonymously, possibly with a slight differnce of emphasis. For a textbook treatment see Univalent Foundations Program (2014).

will be many of these correspondences. The identity type is a set. But then there is nothing further to say about any two such identities, other than that they are the same or different. This type of finite sets therefore has level +1, since identity types are sets, of level 0. And so it goes on. One may have types of infinitely high level.

An important part of Martin-Löf type theory is the notion of a dependent type, denoted $x : A \vdash B(x) : Type$. Here the type $B(x)$ *depends* on an element of A, as in $Days(m)$ for $m : Month$, or $Players(t)$ for $t : Team$. Two useful constructions we can apply to these types are *dependent sum* and *dependent product*.

- Dependent sum: $\sum_{x:A} B(x)$ is the collection of pairs (a, b) with $a : A$ and $b : B(a)$. When A is a set and $B(x)$ is a constant set B, this construction amounts to the product of the sets. Likewise if A is a proposition and $B(x)$ is a constant proposition, B, dependent sum is the conjunction of A and B. In general we can think of this dependent sum as sitting 'fibred' above the base type A, as one might imagine the collection of league players lined up in fibres above their team name.

- Dependent product: $\prod_{x:A} B(x)$, is the collection of functions, f, such that $f(a) : B(a)$. Such a function is picking out one element from each fibre. When A is a set and $B(x)$ is a constant set B, this construction amounts to B^A, the set of functions from A to B. Likewise if A is a proposition and $B(x)$ is a constant proposition, B, dependent product is the implication $A \to B$. In terms of the picture of players in fibres over their teams, an element of the dependent product is a choice of a player from each team, such as $Captain(t)$.

Here then we can see how close this foundational language is to mainstream mathematics and physics. Dependent sum and dependent product correspond respectively to the total space and to the space of sections of fibre bundles, which appear in gauge field theory often in the guise of principle bundles. A fibre bundle is a form of product, but a *twisted* one where as we pass around the base space, a fibre may be identified under a nontrivial equivalence. An easy example is the Moebius strip, where an interval is given a 180 degree twist as it is transported around a circle. Of course, for physics one needs some geometric structure on the base space and total space. We shall see more about this in the section 5 below.

We can now see that these constructions allow us to formulate the quantifiers from first-order logic, at least as defined over types. So consider the case where A is a set, and $B(a)$ is a proposition for each a in A. Perhaps A is the set of animals, and $B(a)$ states that a particular animal, a, breathes. Then an element of the dependent sum is an element a of A and a proof of $B(a)$, so something witnessing a breathing animal. Meanwhile an element of the dependent product is a mapping from each $a : A$ to a proof of $B(a)$. There will only be such a mapping if $B(a)$ is true for each a. If this were the case, we would have a proof of the universal statement 'for all x in A, $B(x)$', in our example, 'All animals breathe.'

Returning to the dependent sum, this is almost expressing the existential quantifier 'there exists x in A such that $B(x)$', except that it's gathering all such a for which $B(a)$ holds, or, in our case, gathering all breathing animals.

As we have seen before in the capped addition of the Boolean truth values in a disjunction, to treat this dependent sum as a proposition, there needs to be a 'truncation' from set to proposition, so that we ask merely whether this set is inhabited, in our case 'Does there exist a breathing animal?' This extra step should be expected as existential quantification resembles forming a long disjunction. That we don't need to adapt for universal quantification tallies with the straightforward form of the product of Boolean values.

What emerges from these lines of thought is that the lower levels of homotopy type theory have contained within them: propositional logic, (typed) predicate logic and a structural type theory. Coming from a tradition of *constructive* type theory, one needs to add classical axioms if these are required, such as various forms of excluded middle.

Before ending this section, I shall introduce a further way to think about dependent sum and dependent product, this time as related via *adjoints*. Adjunctions are category theory's way of dealing with something as close to an inverse as possible when such an inverse does not necessarily exist. Then left and right adjoints are approximations from either side.

Returning to our type theory, given a plain type, A, we can turn any type C into one dependent on A by formulating $x : A \vdash (A \times C)(x) := C$. If A and C are sets, think of lining up a copy of C over every element of A, the product of the two sets projecting down to the first of them, $A \times C \to A$. Now, we can think of approximating an inverse to this process, which would need to send A-dependent types to plain types. Such approximations, or adjoints do indeed exist. Left adjoint to this mapping is dependent sum, and right adjoint is dependent product. The fact that these are adjoints may be rendered as follows, for B a type depending on A:

- $Hom_{\mathcal{C}}(\sum_{x:A} B, C) \cong Hom_{\mathcal{C}/A}(B, A \times C)$
- $Hom_{\mathcal{C}}(C, \prod_{x:A} B) \cong Hom_{\mathcal{C}/A}(A \times C, B)$

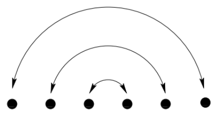
I shall not further explain this formulation except to note that we can see vertical unity at play here here if we return once again to the logic-arithmetic analogy from earlier in this section. Take A, B and C as finite sets, with B fibred over A, which just amounts to a map from B to A, the elements of B fibred above their images in A. Then taking the cardinalities of the two sides of each adjunction above yields further recognisable arithmetic truths:

- $c^{\sum_i b_i} = \prod_i c^{b_i}$
- $(\prod_i b_i)^c = \prod_i (b_i)^c$

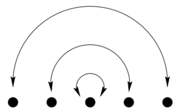
So a pupil being taught that, say, $3^5 \times 7^5 = 21^5$ is being exposed to the shadow of an instance of an important adjunction, which in turn, as with all of the discussion above of dependent sums and products, works for types up and down the hierarchy of n-types. Here *vertical unity* takes on a literal meaning. These n-types in the guise of higher groupoids we turn to next.

4 Groupoids, Identity and Symmetry

As it has turned out, my choice of concept in Corfield (2001) was a good one. Groupoids have flourished in the intervening years, and we can now update their history a little to say more about the hierarchical levels of homotopy type theory. A central case for groupoids that I discussed was their ability to capture 'irregular' quotients. This can be illustrated by a simple example which I first learned from John Baez (see Baez and Dolan 2001). Consider a row of six objects spaced evenly along a line. Inverting the line about the midpoint identifies the objects, yielding three pairs.



There is nothing surprising here. We can take ourselves to be demonstrating the simple mathematical fact that $6/2 = 3$. Whether we take this to be three bound pairs or three singletons matters little. Now let us do the same thing with 5 objects. In this case the central object is unlike the others in that it will be paired with itself.



Classes formed by the identification number three: two pairs and that odd singleton. Here it seems we could be construed as making the evidently false claim that $5/2 = 3$. To repair this it would appear that we should have that special central object count only for one half. Groupoids give us a way to do precisely this.

For any group, in the mathematical sense, acting on a set, we speak of the 'action' of the group, and represent it by the 'action groupoid'. Think of the elements of the set as points or objects, and arrows going from one object to another are labelled by group elements which send the first object to the second. The cardinality of a groupoid is found by summing for each 'component', or connected cluster of objects, the reciprocal of the size of its 'isotopy group', essentially the number of arrows looping from an object to itself. In the case of five objects above, for the object identified to itself in the middle of the row, there are two such arrows. In this case, we have to add $1+1+\frac{1}{2} = \frac{5}{2}$, as desired. Now it is a general result that whatever the group, and whatever its action on a finite set, we will have an arithmetic identity in that the cardinality of the action groupoid is the size of the finite set divided by the order of the group.

For the purposes of physics again we need more geometric structure, but the basic idea of an action by a group of transformations specifying invariance under symmetries is at work. From 'orbifolds', which might result from a Lie group acting unevenly on a manifold, to the objects of field values for gauge field theory, which need to keep track of gauge equivalences, we find we are treating geometric forms of groupoid. If in the latter case instead we take the simple quotient, the plain set of equivalences classes, which amounts to taking gauge symmetries to be redundant, the physics goes wrong in some sense, in that we can't retain a *local* quantum field theory. Furthermore, in the case of gauge equivalence we don't just have one level of arrows between points or objects, but arrows between arrows and so on, representing equivalences between gauge equivalences. And this process continues indefinitely to 'higher' groupoids, needed to capture the higher symmetries of string theory (Schreiber 2013). In conceptual terms the very idea of sameness or identity is being given a much richer interpretation here (Corfield 2003, chap. 10). This is embodied in the form of the identity types we saw in the previous section.

The action groupoid is in fact a manifestation of something we have already seen. Indeed, in homotopy type theory, the way to express that a set, A, is acted on by a group, G, is to write it as a dependent type $* : BG \vdash A(*) : Type$. The type BG is a version of the group G, but where we're taking it to be a single object with looping arrows labelled by each group element. Applying the dependent sum construction from the last section, we find that $\sum_{*:BG} A(*)$ is a type with structure that of the action groupoid, which should indicate to us that the concept is a fundamental one.

Similarly we can form the dependent product, which in this case is composed of the 'fixed points' of the action, those elements of the set left unchanged by all elements of the group. Carrying on in this vein, we soon meet some central constructions from the area known as 'representation theory', a subject taught at the earliest at a senior undergraduate level. So it turns out that important concepts in representation theory are just constructions in homotopy type theory carried out in a context of symmetries, constructions which in more basic settings are familiar logical operations. For example, the type theoretic construction governing the formation of *induced representations*, typically first met as a way of expanding an action of a subgroup to an action of the full group, is very basic when carried out on sets and predicates. For instance, consider the case of the set of dogs sent by the function 'owner' to the set of people, each dog being assigned its (unique) owner. The equivalent of forming an induced representation in this setting is to map any predicate of dogs to a predicate of people, say, 'poodle' is sent to 'owner of some poodle'. Again we find this new foundational language showing us vertical unity between the topic of an advanced undergraduate or graduate course and something the general populace might understand.

With such little conceptual guidance provided by set theory, it's unsurprising that we turn to the naming of concepts for illumination, even if this may be misleading. It seems that there's a thin line between modifying the characterisation of a concept, and devising a new neighbouring concept. Sometimes we just add an epithet to an existing term, as in *imaginary* number or *nondifferentiable function*. Sometimes we modify one, such as *functor* from *function*. However, naming choices are not strictly dictated by how things are best understood. Indeed, there is no reason to take preservation of a name as a sign of great stability. Concerning the natural sciences, Dudley Shapere (1989) once made the excellent point that we can be taken in by the vagaries of naming decisions to see a larger or smaller difference than we ought. He gave the case of 'atom'. Because the name has been kept, we strive to find much greater continuity than for the case of a concept which has changed similarly but been renamed. In fact, in the early nineteenth century 'atom' and 'molecule' were used interchangeably. Had the latter been the only term, a story of the change from 'molecule' then to 'atom' now should involve no more of a change than we tell now from 'atom' then to 'atom' today. It has been argued (Chang 2009) that theories of phlogiston had comparable levels of empirical adequacy to early theories of oxygen, and that judgements as to the justified triumph of the latter are largely due to its having won out historically in the early nineteenth century. We pay the price today by having retained the poorly named oxygen, the supposed 'principle of acidity'.

Over time there's an inclination to absorb the earlier term into a generalisation by dropping the epithet. Then we may need a new term to designate what was special about the original case. Think of 'natural' number or 'differentiable' function. On the other hand, sometimes we resist such terminological change as with 'multivalued function', which is still not an example of a function. This is sometimes referred to as a 'red herring'. In the case of groups, in the early days of their use, Poincaré meant by 'group' what we call *abelian* group. By the time Brandt coined 'groupoid' in the 1920s, the term 'group had the settled usage we see today. He might quite possibly have denoted one as a '*something* group', rather than use the '-oid' suffix. Having opted for 'groupoid', it was always going to be hard to drop the suffix. Now we have higher groupoids and their specialisation to higher groups, there is perhaps a chance to simplify the terminology. But rather than naming policies, the important point of view provided by homotopy type theory reveals the centrality of this cluster of concepts to a conception of mathematics based on identity and structural invariance.

5 Spaces and Duality

In this final section I would like to discuss briefly topics of two further studies of mine (2015, *forthcoming*) which concern the large thematic concepts of space and duality. A common criticism of set theory as a foundational language has been its unnatural treatment of certain concepts, none more so than spatial ones. Where the set theoretic approach can only look to build up a set that behaves like a space from the dust of its points, homotopy type theory can at least build into its basic entities something path-like in the collection of identities between terms. But we must be careful here as

> ...although it is common in homotopy type theory to use terminology borrowed from topology such as "path" and "circle", these words have *a priori* nothing to do with their topological versions *which can also be defined inside of type theory.* (Shulman 2015, p. 3)

Indeed, homotopy type theory is a synthetic theory of structure, not of spaces (Shulman *forthcoming*). This point can be seen clearly from the two versions of circle available in the theory. The native one uses 'higher inductive types' to define a type, S, with a designated element known as its base point, *base*, along with a designated identity element in $Id_S(base, base)$. It is natural to represent this as a point with a loop which may be iterated indefinitely, just as one might wind a string around a circular reel. This type can then be used to do a synthetic form of algebraic topology, such as finding the type of maps from S to itself as an intrinsic form of the fundamental group of the circle, which is then calculated to be isomorphic to the integers (Licata and Shulman 2013).

The second approach is to define a circle as a locus of points in the plane. This is a much more elaborate, and less intrinsic, construction in the type theory, which first requires the construction of the reals, before forming the plane and then the subtype of this which is the circle, in much the way one would ordinarily produce a circle. In the bare type theory there is no connection between these two 'circles'.

Intrinsic ways to define continuity or cohesiveness have therefore been sought, and a solution recently found in the form of the addition of some 'modalities' to homotopy type theory (Schreiber 2013). These are expressed using a string of adjunctions in a way we need only hint at here. The treatment comes from an understanding of a classic case where topological spaces and sets are conceived as related by a series of four adjoint functors. The pair from spaces to sets send a space (a) to the set of its connected components, as though any points connected by a path had been clumped together to a point, and (b) to the set of its points, as though any connectivity between points had been dissolved. The two functors in reverse send a set to the spaces with that underlying set and either the discrete or codiscrete topology, so either one applies no glue, or a maximal glue when forming the spaces. These four functors can be composed in three different ways to yield adjoint functors from spaces to spaces. Abstracting from this situation, a pattern of such a triple of adjunctions gives a general way to express how something spatial is made of parts which hold together cohesively. Now we can relate the two circles above. One can also add in a further triple of adjoints to allow the synthetic expression of many major constructions from differential topology and geometry, to capture smoothness as well as continuity.

As I explain in Corfield (*forthcoming*), however intricate the 'spaces' of current mathematics, we never leave behind the essential idea of constructing them by gluing together specified model spaces. This is recognized by the term 'atlas' used for a collection of 'charts' which cover a manifold, but is an idea present in the recent 'generalized schemes' of Jacob Lurie (2009b) and other spaces of current geometry. This patchwork idea of spaces is perhaps a feature of our everyday understanding of our surrounding space. It seems that when we work out a way to walk across a city, we have something like local maps in our mind which we can look to adjoin to provide us with a feasible route. With the addition of the modalities, homotopy type theory can express this gluing process straightforwardly, as it can the geometric constructions needed to formulate gauge field theories.

Finally, we turn to duality, which in some ways presents an interesting problem for Thesis 1 in that it runs in some sense orthogonal to the n-type hierarchy, and so may fit less well with what we've discussed so far. That it is a pervasive thematic mathematical concept is clear, and that it is superbly well captured by category theory is readily established (Corfield 2015). Moreover, concerning the vertical unity of Thesis 2, duality is certainly rooted in simple, everyday concepts. Take a cube and join the midpoints of its adjacent faces with edges to form an octahedron. Then do the same with an octahedron to form a cube. The eight vertices, twelve edges and six faces of the cube are dual to the eight faces, twelves edges and six vertices of the octohedron. We can likewise pair the dodecahedron with the icosahedron and the tetrahedron with itself to complete the platonic solids. This is an example of what is sometimes called *formal* or *abstract duality*, where there is an exchange of kinds of entity and relations between them.

Important pieces of mathematics then arise from relating this formal duality with *concrete* duality, which arises from mappings which assign a value to pairs of entities. Consider a simple truth theoretic pairing, where for an actor and a film we assign a truth value depending on whether or not they appear in the film. For a typical range of actors and films it will be the case that the list of films in which an actor appears determines that actor and the list of actors in a film determines that film. In mathematics we often find *perfect* pairings between two collections where it is possible to reconstruct an element of the first collection from the values resulting by pairing it with elements of the second collection, and *vice versa*. The inner product of a vector space is an example of this situation where a vector is determined by the values given by its inner product with a complete set of linearly independent vectors. The duality between platonic solids can be reconsidered in this setting, relating formal and concrete dualities.

While vertical unity is apparent within the topic of duality, stretching from these simple examples to the very forefront of mathematical research, the issue of how it relates to varieties of type theory is not so clear yet. Naturally, one can reconstruct specific examples of duality in these languages, just as one can reconstruct any mathematics within set theory, but one might expect general aspects of such a fundamental concept to be expressible synthetically. However, the research front is moving fast here. It appears that there is a *linear* version of homotopy type theory, which captures some of the duality structures found in the mathematics of quantum physics (Schreiber 2014), and elsewhere proposals for an 'adjoint logic'. It will be interesting to see how this work plays itself out.

6 Conclusion

In this chapter I set out to make the case for two theses

1. This new foundational language, homotopy type theory, provides an important perspective from which to understand varieties of mathematical concept.

2. Mathematics displays *vertical unity*, so that concepts met in elementary and in research level mathematics are related.

As I remarked, these theses are related in that the new language is often able to reveal the commonality between elementary and advanced concepts as arising from a single form of construction within the type theory.

At the same time, there is no claim on my part that every mathematical concept finds its simplest expression within homotopy type theory. However, we should see here the possibility of assessing constructions as to whether they are intrinsic to all type theories, intrinsic to specific type theories (such as the homotopy, modal homotopy and linear varieties), or as requiring complex definitions within a specific type theory. The product construction, manifesting as set product and conjunction of propositions, is a good example of a very basic form of concept.

Eventually we may find some form of complexity measure for a concept or result which adds the complexity of a formal framework to that of the construction of the concept or result within that framework. We should expect that what is complicated according to one type theory may be approached more intrinsically within another. Good examples of this are Stokes' theorem and Noether's theorem which appear highly complex in set theory, but which become near logical results when expressed in modal homotopy type theory (Schreiber *forthcoming*).

In that homotopy type theory and other type theories fare well at displaying the reasons for vertical unity, I believe we can take this as a good indication of their power.

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